PHIL 114 Final 2010

Tarski's World

You may use the predicates you are accustomed to, e.g., "Tet(x), Dodec(x), Large (x), Medium(x), LeftOf(x,y), etc. and any letters (a, b, c, etc.) for names.

1. Say in FOL "a is neither large nor small only if a is medium"
2. Say in QPL "Every cube and every dodecahedron is small"
3. Construct a full truth table for the expression "(Cube(b) ∧ LeftOf(b,c)) → Small(b)".
4. Construct a full truth table for the expression "∃x¬Cube(x) → ¬∀xCube(x)".
5. Construct a formal proof of "Cube(a) → RightOf(a,b)" from the premise "RightOf(a,b) v (Tet(a) v Dodec(a))". You may use AnaCon, and FOCon and TautCon ONLY ON LITERALS! (e.g., inferring ¬Large(a) from Small(a) is an acceptable use of AnaCon, but more complicated inferences must be done with the intro and elim rules).

(Peano) Arithmetic

Consider the following propositions, where "0","1", "2", etc. are names for the integers, "+" is defined in the usual way as addition, "x" as multiplication and "=" is the identity relation. Let us assume the following as basic axioms of arithmetic:

A1. $\forall x \forall y(x + 1 = y + 1 \rightarrow x = y)$

A2. $\forall x(x + 1 \neq 0)$

A3. $0 + 1 = 1$

A4. $\forall x(x + 0 = x)$

A5. $\forall x \forall y[x + (y + 1) = (x + y) + 1]$

A6. $\forall x(x \times 0 = 0)$

A7. $\forall x \forall y[x \times (y + 1) = (x \times y) + x]$

AS. $Q(0) \wedge \forall x(Q(x) \rightarrow Q(x + 1))] \rightarrow \forall x Q(x)$

("AS" is a so-called "axiom-scheme" which enables us to do mathematical induction. It is taken to be true of any well-formed formula Q which involves x.)

6. Construct a full truth table for the expression "1=1↔ 2=2"
7. Even though your truth table doesn't show it, what must the truth value of the expression in the previous problem be?
8. Consider the expression "If 1=2, then P". Does its truth value change depending on the truth value of P? Explain your answer.
9. From Peano's Axioms (A1-AS) prove $\forall x(0 + x = x)$. In particular the premises you will need are: $\forall x(x + 0 = x)$, $\forall x \forall y[x + (y + 1) = (x + y) + 1]$, $0 + 1 = 1$, $[Q(0) \wedge \forall x(Q(x) \rightarrow Q(x + 1))] \rightarrow \forall x Q(x)$, where Q(x) is (0 + x = x).
10. From Peano's Axioms (A1-AS) prove $\forall x(1 \times x = x)$. In particular the premises you will need are: $\forall x(x \times 0 = 0)$, $\forall x \forall y[x \times (y + 1) = (x \times y) + x]$, $[Q(0) \wedge \forall x(Q(x) \rightarrow Q(x + 1))] \rightarrow \forall x Q(x)$, where Q(x) is $(1 \times x = x)$.

Syntactic Proofs

All letters P, Q, R, etc. stand for arbitrary strings of symbols that have been well formed. In each case your job is to present a correct formal proof of the conclusion using only introduction and elimination rules for the connectives. You are *not* allowed to use AnaCon, TautCon, etc. THERE IS PARTIAL CREDIT FOR PROOF SKELETONS!

11. Premises: $(P \vee Q) \rightarrow R$, $\neg T \rightarrow \neg R$. Conclusion: $P \rightarrow T$
12. Premise: $\neg\forall x \mathrm{Cube}(x)$. Conclusion: $\exists x \neg \mathrm{Cube}(x)$.
13. Premise: $\exists x \forall y\, L(x,y)$. Conclusion: $\forall y\, \exists x\, L(x,y)$
14. Premises: $\forall y\, \exists x\, (x < y)$, $\exists x(x < a) \rightarrow \forall z(z < a)$ Conclusion: $\exists x \forall y\, (y < x)$.
15. Conclusion: $\forall z((P(z) \rightarrow Q(z)) \rightarrow \neg((P(z) \wedge \neg(Q(z)))$
16. Premise: $\forall a_s \forall b_s (\forall x(x \in a_s \leftrightarrow x \in b_s) \rightarrow a_s = b_s)$.
    Conclusion: $\exists b_s \forall a_s (\forall x(x \in a_s \leftrightarrow x \in b_s) \rightarrow a_s = b_s)$.
    (Here I use a_s and b_s to emphasize that these are *not* constants, but variables. The quantifier $\forall a_s$ quantifies over sets as opposed to over elements of sets. The quantifier $\forall x$ quantifies over elements of sets.)

Modal Notions, Meta-Concepts, and Truth-Functionality

17. Give a *clear* example that shows that the one place operator "It is unlikely that______ " is not a truth functional operator. You may use any propositions you like. (The operator "It is unlikely that ____" is the normal one used in sentences such as "It is unlikely that my logic teacher will do cartwheels down the room" Also briefly explain how your example shows non-truth functionality. It may help to compare this operator with the operator "It is not the case that______". Be as explicit as possible.
18. Using a truth table, write a few sentences that explain why the proposition a=a is not tautologically (i.e. truth table) necessary but is logically necessary.
19. Write a few sentences that explain why the proposition "Everything is a P only if something is a P" is first order necessary (i.e. a first order validity) even though it is not tautologically necessary. A truth table may be helpful.
20. Write a few sentences that explain why the proposition "Nothing is taller than itself" is logically necessary even though it is not first order necessary nor tautologically necessary.
21. To say that QPL is complete roughly means that with QPL you can prove everything that is true. Likewise if Peano Arithmetic were complete you could prove all arithmetic truths with it. Now consider the sentence "This sentence is not provable", and call it *P*. Imagine that someone very clever wrote *P* down in our logical symbols, making use of Peano's Axioms in clever ways. If *P* were true, could we prove *P*? What does *P* suggest about the completeness of Peano Arithmetic (if *P* could be expressed in Peano Arithmetic)?

Bonus:

Use Peano's Axioms and the Introduction and Elimination rules for the various operators we have to prove the commutivity of addition, namely, $\forall x \forall y(x + y = y + x)$. The tough part of this is using the Axiom Scheme (AS) in the right ways!